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A multidimensional version of Turán's lemma

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Abstract

In this article we provide a multidimensional version of Nazarov's extension of Turán's lemma—a result in which the uniform norm of a complex-valued polynomial, p, on the unit circle $\mathbb{T}$ is compared with the uniform norm of p on any measurable subset of $\mathbb{T}$. If we let $\mathbb{T}^n := \mathbb{T} \times \cdots \times \mathbb{T}$ represent the distinguished boundary of the polydisk $D^n := D \times \cdots \times D$ for $n \in \mathbb{N}$ and D the open unit disk then, as in the one dimensional case, the constant which relates the uniform norm of p on $\mathbb{T}^n$ to the uniform norm of p on any measurable subset E of $\mathbb{T}^n$ depends on the order of p and the measure of the set E.

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1. Introduction

Let $\mathbb{T}$ denote the unit circle and D the open unit disk. Let $p : \mathbb{T} \to \mathbb{C}$ be the polynomial defined by $p(z) := \sum_{k=0}^{m} c_k z^{r_k}$ for $c_k \in \mathbb{C}\backslash\{0\}$ and $r_0 < r_1 < \cdots < r_m \in \mathbb{Z}$, so that the order of p, i.e., the number of non-zero coefficients of p, is $m + 1$. Also, for any subset S of $\mathbb{T}$ define the norm of p on S to be $\|p\|_S := \sup_{z\in S} |p(z)|$. Let λ represent the normalized Lebesgue measure on $\mathbb{T}$.

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Nazarov's extension of Turán's lemma is the following:

Theorem 1 (*Nazarov [2,3]*). *Let E be any measurable subset of $\mathbb{T}$ with $\lambda(E) > 0$. Then*

$$\|p\|_{\mathbb{T}} \leqslant \left(\frac{14}{\lambda(E)}\right)^m \|p\|_E.$$

In contrast to Turán, who assumed E to be an arc of $\mathbb{T}$, Nazarov generalized Turán's original result [5] by considering the set E to be any measurable subset of $\mathbb{T}$.

Theorem 1 is interesting in that we can gain information about the maximum modulus of p on $\mathbb{T}$ by only knowing the maximum modulus of p on $E \subseteq \mathbb{T}$. Of course, if the measure of E is small, the constant term $\left(\frac{14}{\lambda(E)}\right)^m$ is large, thus making the upper bound less precise; we should not expect to gain explicit bounds for $\|p\|_{\mathbb{T}}$ if we are only given information about the maximum value of p on a very small subset of $\mathbb{T}$.

It is also interesting to note that the constant $\left(\frac{14}{\lambda(E)}\right)^m$ depends on the order of p rather than the degree of p. It is easier to prove this lemma in the case that p is of degree m, rather than of order $m + 1$ [1].

2. Multidimensional result

In this paper, we would like to prove a multidimensional version of Nazarov's extension of Turán's lemma. Before doing so, we must introduce the following standard multivariate notation [4]: let $\mathbb{T}^n := \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}$ be the distinguished boundary of the polydisk $D^n := D \times D \times \cdots \times D$ and define $\mathbf{z} \in \mathbb{T}^n$ by $\mathbf{z} := (z_1, \ldots, z_n) \in \mathbb{T}^n$.

In order to define a polynomial $p : \mathbb{T}^n \to \mathbb{C}$, let the index k of the polynomial be given by $k = (k_1, \ldots, k_n)$ for $0 \leqslant k_i \leqslant m_i$, $m_i \in \mathbb{Z}^+$. Let the exponents r_k of $\mathbf{z}$ be defined by $r_k = (r_{1,k_1}, r_{2,k_2}, \ldots, r_{n,k_n})$ where $r_{i,0} < r_{i,1} < \cdots < r_{i,m_i} \in \mathbb{Z}$. Define $\mathbf{z}^{r_k} := {z_1}^{r_{1,k_1}} {z_2}^{r_{2,k_2}} \cdots {z_n}^{r_{n,k_n}}$ and let $c(k)$ be non-zero coefficients in $\mathbb{C}$. We define the polynomial $p : \mathbb{T}^n \to \mathbb{C}$ by $p(\mathbf{z}) := \sum_k c(k)\mathbf{z}^{r_k}$. So, for example, in the case that $n = 2$, p has the form

$$p(z_1, z_2) = \sum_{k_2=0}^{m_2} \sum_{k_1=0}^{m_1} c_{k_1,k_2} {z_1}^{r_{1,k_1}} {z_2}^{r_{2,k_2}}.$$

Let λ be the normalized Lebesgue measure on $\mathbb{T}^n$ and define the uniform norm of p on any subset S of $\mathbb{T}^n$ to be $\|p\|_S = \sup_{\mathbf{z} \in S} |p(\mathbf{z})|$. The following is a multidimensional version of Nazarov's extension of Turán's lemma:

Theorem 2. *Let $p(\mathbf{z}) := \sum_k c(k)\mathbf{z}^{r_k}$. If $E \subseteq \mathbb{T}^n$ is a measurable set with $\lambda(E) > 0$, then*

$$\|p\|_{\mathbb{T}^n} \leqslant \left(\frac{14n}{\lambda(E)}\right)^{m_1+m_2+\cdots+m_n} \|p\|_E.$$

Proof. We will proceed using induction on the dimension n. Due to Theorem 1, it has already been shown that the base case, $n = 1$, is true. Assume that the theorem holds for all $n \leqslant j$ for some $j \in \mathbb{Z}^+$. We wish to prove that the result holds in the case $n = j + 1$.

Assume $E \subseteq \mathbb{T}^{j+1}$ is a measurable set with positive measure and let $p : \mathbb{T}^{j+1} \to \mathbb{C}$ be defined by $p(\mathbf{z}) := \sum_k c(k)\mathbf{z}^{r_k}$. For any $z \in \mathbb{T}$ define $E_z := \{(z_1, \ldots, z_j) \in \mathbb{T}^j : (z_1, \ldots, z_j, z) \in E\}$.

Define $B := \{z \in \mathbb{T} : \lambda(E_z) \geqslant \frac{\lambda(E)}{C}\}$ where $C := \frac{j+1}{j}$, so that if $z \in B$, then the measure of E_z is bounded away from zero, and say that $\|p\|_{\mathbb{T}^{j+1}} := |p(w_1, \ldots, w_{j+1})|$ for some $(w_1, \ldots, w_{j+1}) \in \mathbb{T}^{j+1}$. Then by applying Theorem 1 to the polynomial $p(w_1, \ldots, w_j, z)$ and the set B we get that

$$\|p\|_{\mathbb{T}^{j+1}} \leqslant \left(\frac{14}{\lambda(B)}\right)^{m_{j+1}} \sup_{z \in B} |p(w_1, \ldots, w_j, z)|.$$

Therefore, given any $\varepsilon > 0$, there exists some $z_* \in B$ for which

$$\|p\|_{\mathbb{T}^{j+1}} \leqslant \left(\frac{14}{\lambda(B)}\right)^{m_{j+1}} (|p(w_1, \ldots, w_j, z_*)| + \varepsilon).$$

By assumption, we may also apply Theorem 2 to the polynomial $p(z_1, \ldots, z_j, z_*)$ and the set $E_{z_*} \subseteq \mathbb{T}^j$. This gives us that

$$\begin{aligned} \sup_{(z_1,\ldots,z_j) \in \mathbb{T}^j} |p(z_1, \ldots, z_j, z_*)| &\leqslant \left(\frac{14j}{\lambda(E_{z_*})}\right)^{m_1+m_2+\cdots+m_j} \sup_{(z_1,\ldots,z_j) \in E_{z_*}} |p(z_1, \ldots, z_j, z_*)| \\ &\leqslant \left(\frac{14j}{\lambda(E_{z_*})}\right)^{m_1+m_2+\cdots+m_j} \|p\|_E \end{aligned}$$

since $(z_1, \ldots, z_j, z_*) \in E$ for $(z_1, \ldots, z_j) \in E_{z_*}$.

By combining these two results and taking the limit as ε approaches zero, we have

$$\|p\|_{\mathbb{T}^{j+1}} \leqslant \left(\frac{14}{\lambda(B)}\right)^{m_{j+1}} \left(\frac{14j}{\lambda(E_{z_*})}\right)^{m_1+m_2+\cdots+m_j} \|p\|_E. \quad (1)$$

Therefore, in order to complete the proof we must find lower bounds for $\lambda(E_{z_*})$ and $\lambda(B)$ in terms of $\lambda(E)$.

Note

$$\lambda(E_{z_*}) \geqslant \frac{\lambda(E)}{C}$$

since $z_* \in B$. Let B^C denote the complement of the set B. By considering the geometry of the set E, and the maximum values that $\lambda(E_z)$ can take for any $z \in B$ and for any $z \in B^C$ respectively, we get that

$$\begin{aligned} \lambda(E) &\leqslant \lambda(B)(1)^j + \lambda(B^C) \cdot \frac{\lambda(E)}{C} \\ &\leqslant \lambda(B) + \frac{\lambda(E)}{C}. \end{aligned}$$

This implies that

$$\lambda(B) \geqslant \lambda(E) \left(\frac{C-1}{C}\right).$$

Plugging in these estimates for $\lambda(E_{z_*})$ and $\lambda(B)$ in (1) and using the fact that $C = \frac{j+1}{j}$ gives us

$$\|p\|_{\mathbb{T}^{j+1}} \leqslant \left(\frac{14(j+1)}{\lambda(E)}\right)^{m_1+m_2+\cdots+m_{j+1}} \|p\|_E,$$

which completes the proof. □

Surprisingly, in contrast to the univariate version of this theorem, in higher dimensions the exponent $m_1 + m_2 + \cdots + m_n$ on the constant relating the two norms of p, is much smaller than the order $(m_1 + 1)(m_2 + 1) \cdots (m_n + 1)$ of the polynomial.

References

[1] P. Borwein, T. Erdélyi, Polynomials and Polynomial Inequalities, Springer, Berlin, 1995.
[2] F. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type, Algebra i Analiz 5 (4) (1993) 3–66; translation in St. Petersburg Math. J. 5 (4) (1994) 663–717.
[3] F. Nazarov, Complete Version of Turán's Lemma for Trigonometric Polynomials on the Unit Circumference (English Summary), Complex Analysis, Operators, and Related Topics, Birkhäuser, Basel, 2000, pp. 239–246.
[4] W. Rudin, Function Theory in Polydiscs, Benjamin, New York, 1969.
[5] P. Turán, On a New Method in Analysis and its Applications, Wiley-Interscience, New York, 1984.